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Origamics involving circles

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Abstract

Origamics can be enriched and more friendly to mathematics classes if we consider certain circles in a folded figure.

1 Introduction

Traditional Japanese origami is well known as to make an attractive three dimensional figure such as a crane by folding a sheet of square paper. On the other hand, Kazuo Haga has been lecturing school teachers his mathematical theory of paper folding under the name of *origamics* [2, 3, 4]. Its aim is not to make three dimensional figures but to explore mathematics through paper folding. Thereby it is a good instructional material in mathematics classes. According to [2, 3, 4], circles (especially incircles and circumcircles of triangles, quadrangles, etc.) are not considered explicitly in origamics, though they are essential in high school geometry. Contrarily in Japanese traditional mathematics called wasan, there are an enormous number of geometric problems involving incircles. Indeed there are problems in wasan geometry considering folded figure involving incircles of triangles. They suggest that it would be a nice idea to consider circles in origamics. In this article we show that origamics could be enriched and more friendly to mathematics classes, if we consider some circles in folded figures.

2 Haga's theorem

Let us consider a sheet of square paper $ABCD$ with side length of s and a point E on the side DA. We fold the paper so that the corner C coincides with E and the side BC is carried into $B'E$, which intersects the side AB at a point F (see Figure 1). We call this Haga 's fold of the first kind. He discovered if E is the midpoint of AD, then F divides AB in the ratio $2:1$ internally (Haga's first theorem). Also if F is the midpoint of AB, then E divides AD in the ratio $2:1$ internally (Haga's third theorem).

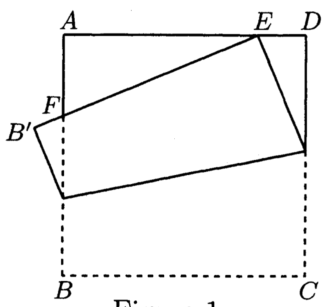

Figure 1.

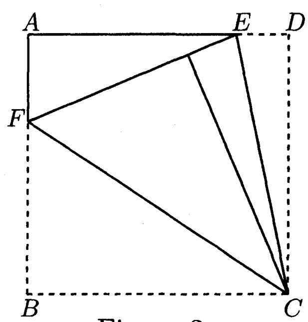

Figure 2.

Let F be a point on the side AB such that the reflection of B in the line CF coincides with the refection of D in the line CE (see Figure 2). If we fold the paper with crease lines CE and CF, we call this Haga's fold of the second kind. He discovered if F is the midpoint of AB, then E divides AD in the ratio $2:1$ internally (Haga's second theorem). Haga's three theorems are the most popular results in origamics.

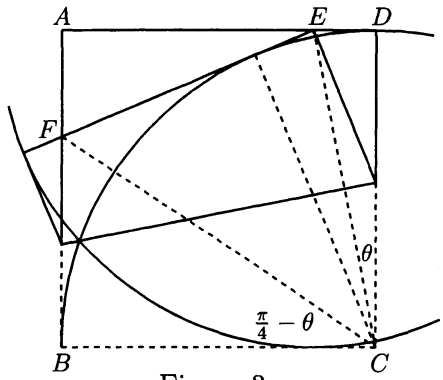

Figure 3.

Haga pointed out that his fold of the first kind derived from the fold of the second kind, and vice versa. In fact for the fold of the first kind, the reflection of B in the line CF coincides with the reflection of D in the line CE. For the reflection of the circle with center E touching the side BC in the crease line is the circle with center C touching the side EF, which also touches AB and DA at B and D, respectively [5] (see Figure 3). This implies

$$|EF| = |DE| + |BF|. \tag{1}$$

Let $\theta = \angle DCE$ and $t = \tan\theta$. Haga's results are unified.

Theorem 1 ([7]). *The relation $|AF|/|BF| = 2|DE|/|AE|$ holds for Haga's fold of the two kinds.*

Proof. We get $|DE| = st$ and $|AE| = s(1-t)$. This implies $|DE|/|AE| = t/(1-t)$. While $\angle BCF = \pi/4 - \theta$ leads $|BF| = s\tan(\pi/4-\theta) = s(1-t)/(1+t)$ and $|AF| = s - |BF| = 2st/(1+t)$. Hence we get $|AF|/|BF| = 2t/(1-t)$. The theorem is now proved. □

Figure 1 has several interesting properties [1], [5] and [8]. Indeed six problems are proposed from this figure in [1].

3 Sangaku geometry, incircle and circumcircle

In Edo era, there was a unique mathematical custom in Japan. When people found nice problems, they wrote their problems on a framed wooden board, which was dedicated to a shrine or a temple. The board is called a *sangaku.* It was also a means to publish a discovery or to propose a problem. Most such problems were geometric and the figure were beautifully drawn in color. The uniqueness is not only the custom, but also the contents of the problems. Ordinary triangle geometry mainly concerns the properties of "one" triangle. On the contrary, sangaku problems concern about some relationship arising from a number of mixed elementary figures like circles, triangles, squares, etc.

Let r be the inradius of the triangle AFE for Haga's fold of the two kinds. The following theorem is taken from a sangaku in Fukushima prefecture (see Figure 4).

Theorem 2. *$|B'F| = r$ holds for Haga's fold of the first kind.*

Proof. By (1), we get

$$\begin{aligned} r &= \frac{|AE| + |AF| - |EF|}{2} = \frac{(s - |DE|) + (s - |BF|) - (s - |B'F|)}{2} \\ &= \frac{s - |DE| - |BF| + |B'F|}{2} = \frac{s - |EF| + |B'F|}{2} = \frac{2|B'F|}{2} = |B'F|. \end{aligned}$$

□

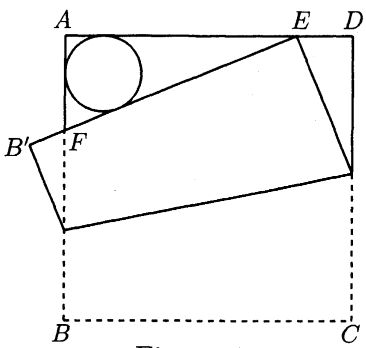

Figure 4.

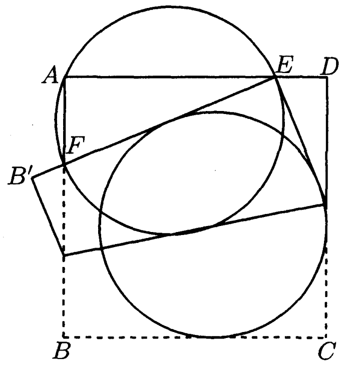

Figure 5.

The above proof is essentially the same to that of [5]. Figure 4 is used in the front cover of the journal on which [8] appeared. As in Figure 4, figures of many sangaku problems were made by adding incircles of triangle, quadrangle, etc. However sangaku problems involving circumcircles are in a minority. Indeed as far as we know, there is no sangaku problems considering the circumcircle of the triangle AFE (see Figure 5).

Theorem 3 ([6]). *The circumcircle of the triangle AFE is congruent to the incircle of the triangle formed by the lines BC, CD and EF for Haga's fold of the two kinds.*

Proof. By the proof of Theorem 1, $|EA| = s(1-t)$, $|AF| = 2ts/(1+t)$. Hence $|EF| = (1+t^2)s/(1+t)$. Therefore the circumradius of AFE is

$$\frac{(1+t^2)}{2(1+t)}s. \tag{2}$$

Also

$$r = |B'F| = s - |EF| = \frac{(1-t)t}{1+t}s.$$

Let h be the distance between A and EF. Then $h = |EA||AF|/|EF| = 2t(1-t)s/(1+t^2)$. While the distance between C and EF is s. Since AEF and the triangle formed by BC, CD and EF are similar, the inradius of the latter is rs/h, which also equals (2). □

4 Some other properties

In the previous section we consider the incircle and the circumcircle of the triangle AFE for Haga's fold of the two kinds. In this section we give several properties related to those circles without proofs. Since most of the properties can be proved using elementary trigonometric functions, they can be used in high school mathematics classes. Let R be the circumradius of the triangle AEF. It is expressed by (2).

Theorem 4. *The following statements hold.*
(i) $2R + r = s$.
(ii) *R takes the minimal value $s \tan \pi/8$ and r takes the maximal value $s \tan^2 \pi/8$ when $\theta = \pi/8$.*

Let K be a point on the line BC such that B lies between K and C, and let L be a point lying in the square $ABCD$. The angle between the lines KL and BC is denoted by ϕ. The inradius of the triangle made by the lines DA, AB (resp. BC, CD) and KL is denoted by r_1 (resp. r_2) (see Figure 6). Also the radius of the excircle of the triangle made by the lines DA, AB (resp. BC, CD) and KL touching KL from the side opposite to A (resp. C) is denoted by r'_1 (resp. r'_2) (see Figure 7).

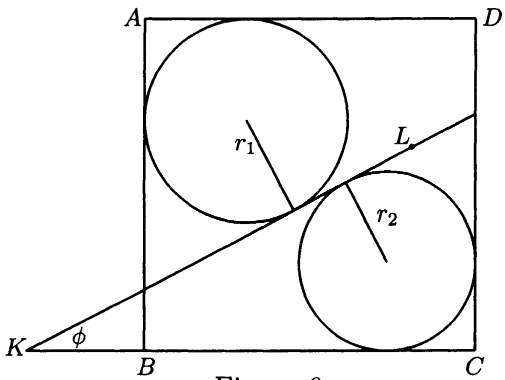

Figure 6.

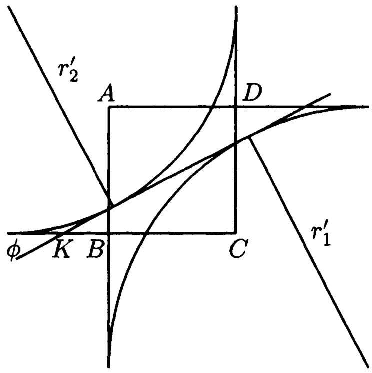

Figure 7.

Theorem 5. *The following statements hold.*
(i) *The sum $r_1 + r_2$ is depend only on ϕ, and*

$$\frac{1}{r_1 + r_2} = \frac{1}{s}\left(1 + \frac{1}{\cos\phi + \sin\phi}\right). \tag{3}$$

(ii) *The sum $r'_1 + r'_2$ is depend only on ϕ, and*

$$\frac{1}{r'_1 + r'_2} = \frac{1}{s}\left(1 - \frac{1}{\cos\phi + \sin\phi}\right).$$

(iii)

$$\frac{1}{r_1 + r_2} + \frac{1}{r'_1 + r'_2} = \frac{2}{s}.$$

Theorem 5 yields several results (see Figures 8, 9 and 10).

Corollary 1. *If l is the perpendicular to KL passing through L, the following statements hold.*
(i) *If s_1 (resp. s_2) is the inradius of the triangle made by the lines CD, DA (resp. AB, BC) and l, then $r_1 + r_2 = s_1 + s_2$ holds.*
(ii) *If s'_1 (resp. s'_2) is the radius of the excircle of the triangle made by CD, DA (resp. AB, BC) and l touching l from the side opposite to D (resp. B), then $r'_1 + r'_2 = s'_1 + s'_2$ holds.*

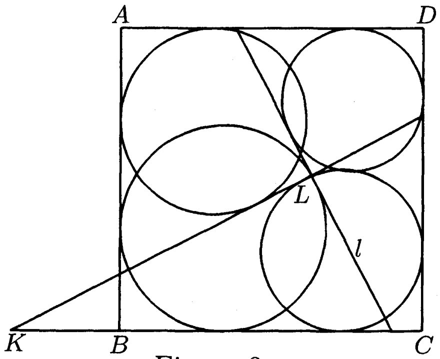

Figure 8.

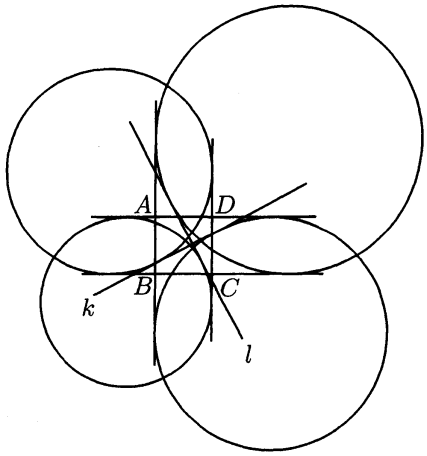

Figure 9.

Corollary 2. *If I is the foot of perpendicular from C to EF, then the sum of the inradii of the kites $CIFB$ and $CDEI$ equals $R + r$ for Haga's fold of the two kinds. If the angle between the lines EF and BC equals ϕ, then $1/(R + r)$ is expressed by the right side of* (3).

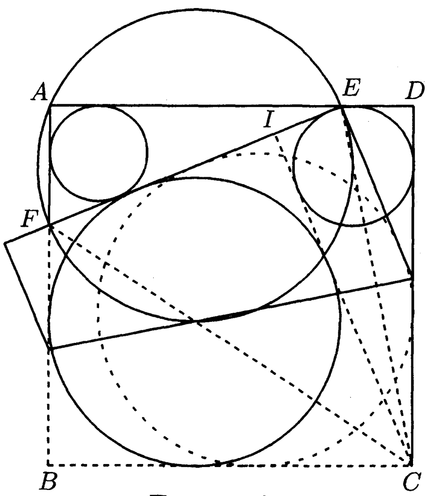

Figure 10.

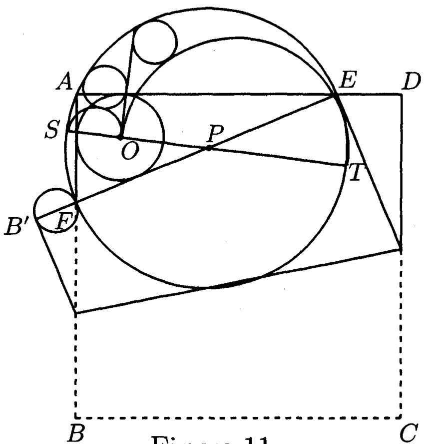

Figure 11.

For a point O on the segment ST, let us consider semicircles with diameters SO, TO and ST constructed on the same side of the line ST. We denote them by (SO), (TO) and (ST), respectively. The area surrounded by the three semicircles is called an arbelos generated by the points S, T and O. Circles of radius equal to the half the harmonic mean of the radii of (SO) and (TO) are called Archimedean circles of the arbelos. The incircle of the curvilinear triangle made by (ST), the perpendicular to ST passing through O and one of (SO) and (TO) is Archimedean (see Figure 11).

Theorem 6. *Let O and P be the incenter and the circumcenter of the triangle AFE for the Haga's fold of the first kind. If the line OP intersects the circumcircle at points S and T, then the circle with a diameter $B'F$ is an Archimedean circle of the arbelos generated by the points S, T and O.*

5 Conclusion

We show that origamics can be enriched and more friendly to mathematics classes if we consider certain circles in folded figures. It seems that origami and wasan should be used in mathematics classes in Japan, since they are traditional Japanese cultural heritages.

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